

# Bosonic clouds with attractive interaction beyond the local interaction approximation

L. Reatto<sup>1</sup>, A. Parola<sup>1,2</sup> and L. Salasnich<sup>1</sup>

<sup>1</sup>*INFM and Dipartimento di Fisica, Università di Milano,  
Via Celoria 16, 20133 Milano, Italy*

<sup>2</sup>*Istituto di Scienze Fisiche, Università di Milano,  
Via Lucini 3, Como, Italy*

*We study the properties of a Bose–Einstein condensed cloud of atoms with negative scattering length confined in a harmonic trap. When a realistic non local (finite range) effective interaction is taken into account, we find that, besides the known low density metastable solution, a new branch of Bose condensate appears at higher density. This state is self-bound but its density can be quite low if the number  $N$  of atoms is not too big. The transition between the two classes of solutions as a function of  $N$  can be either sharp or smooth according to the ratio between the range of the attractive interaction and the length of the trap. A tight trap leads to a smooth transition. In addition to the energy and the shape of the cloud we study also the dynamics of the system. In particular, we study the frequencies of collective oscillation of the Bose condensate as a function of the number of atoms both in the local and in the non local case. Moreover, we consider the dynamics of the cloud when the external trap is switched off.*

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## 1. INTRODUCTION

In the standard treatment of Bosonic alkali atoms in a trap, a local form (i.e. momentum independent) is assumed as effective interatomic interaction<sup>1</sup>. This can not be completely correct when the scattering cross section has a significant momentum dependence already at very low momenta<sup>2</sup>. This is the case of  $^7\text{Li}$ , a particularly interesting case due to its negative scattering length. This momentum dependence implies<sup>3</sup> that the

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effective interaction is non local changing from attractive to repulsive at a characteristic range  $r_e$ .

Recently we have studied<sup>4</sup> the ground state of  ${}^7\text{Li}$  atoms with a non local interaction in a harmonic trap  $U_{\text{ext}}(\mathbf{r}) = \frac{1}{2}m\omega_0^2 r^2$  and we have shown the existence of a new branch of states intermediate in density between the known very dilute state and the collapsed high density state. Here we study how the non locality affects the dynamics of the system. We assume that the attractive potential has a finite range  $r_e$  and in addition we allow for the presence of a repulsive contribution which is modeled as a *local* positive term defined by a “high energy” scattering length  $a_R > 0$ . The effective interaction is then written in the following form<sup>4</sup>:

$$v_{\text{eff}}(k) = \frac{4\pi\hbar^2}{m} [a_R + (a_T - a_R) f(kr_e)] , \quad (1)$$

where  $f(x) = (1+x^2)^{-1}$ . We use interaction parameters appropriate for  ${}^7\text{Li}$ :  $a_T = -27 a_B$ ,  $r_e = 10^3 a_B$  and  $a_R = 6.6 a_B$  (where  $a_B$  is the Bohr radius).

## 2. CONDENSATE GROUND STATE

The ground state wavefunction of a cloud of  $N$  atoms is determined by minimizing the Gross–Pitaevskii (GP) functional  $\mathcal{E}[\Psi]$ , where  $\Psi(\mathbf{r})$  is the wavefunction of the condensate<sup>5</sup>. In the ground state  $\Psi(\mathbf{r})$  is positive definite and spherically symmetric.

As a first step, we discuss an approximate variational approach to this problem which already shows the main features of the exact solution. As a trial wavefunction we choose a Gaussian with a single variational parameter  $\sigma$  (standard deviation) which defines the size of the cloud in units of the harmonic oscillator length  $a_H = (\hbar/(m\omega_0))^{1/2}$ . With this choice, the energy  $\mathcal{E}(\sigma)$  can be analytically expressed in terms of elementary functions. The extrema of  $\mathcal{E}(\sigma)$  are obtained as solutions of an algebraic equation, which gives the number of bosons as a function of the size  $\sigma_0$  of the cloud:

$$N = (1 - \sigma_0^4) \left[ -\gamma_R \sigma_0^{-1} - \frac{1}{3} \tau_1 \sigma_0 + \frac{2}{3\sqrt{\pi}} \chi \tau_2 \sigma_0^3 - \frac{2}{3} \chi^2 \tau_2 \sigma_0^4 g(\chi \sigma_0) \right]^{-1} , \quad (2)$$

where  $\gamma_R = (2/\pi)^{1/2} a_R/a_H$ ,  $\tau_1 = (2/\pi)^{1/2} a_H(a_T - a_R)/r_e^2$ ,  $\chi = 2^{-1/2} a_H/r_e$ ,  $\tau_2 = a_H^2(a_T - a_R)/r_e^3$  and  $g(x) = \text{erfc}(x) \exp(x^2)$  with  $\text{erfc}(x) = 1 - \text{erf}(x)$  the complementary error function. This equation has either one or three positive roots depending on the parameters and on number  $N$  of atoms in the cloud. When three solutions are present, the intermediate one represents an unstable state (i.e. a local maximum of the energy) while the other two respectively describe a low density metastable solution and a minimum which

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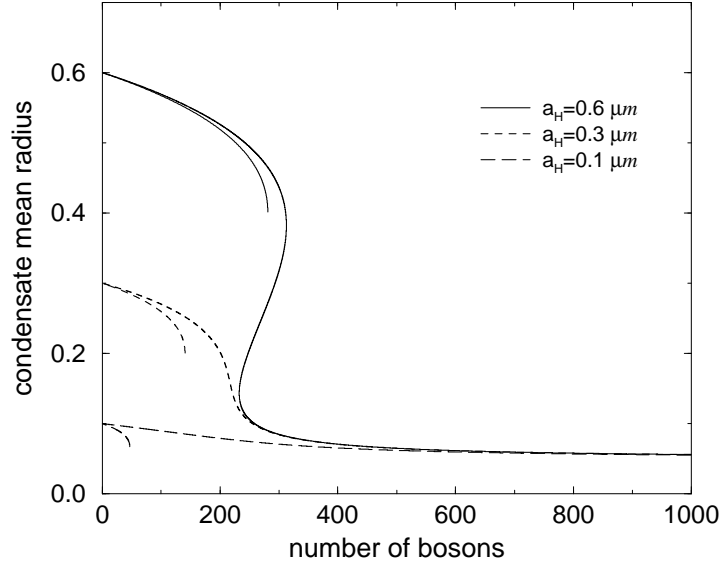


Fig. 1. Mean radius of the condensate, in micron, as a function of the number of bosons for 3 different traps. The lines with a end point represent the results with local interaction.

represents the stable solution within GP approximation. The variational results for three typical trap sizes are shown in Fig. 1, where the mean radius is plotted as a function of  $N$ . For comparison, we also show the radius of the cloud when a local interaction is assumed ( $r_e = 0$  in Eq. (1)). In this case there is a critical number  $N_c \simeq 0.67a_H/a_T$  of bosons beyond which the cloud collapses<sup>1</sup>.

We have also computed the exact solution of the GP equation, obtained by numerical integration of the corresponding self-consistent Schrödinger equation. The variational approach is always very close to the exact solution<sup>4</sup>. The effects of non-locality are always important for very tight traps while for larger traps non-locality is important just when the radius of the cloud rapidly drops for increasing  $N$ . This “transition” is discontinuous for large traps, where the reentrant behavior of the curve shows the presence of an unstable branch. By reducing the trap size, however, this discontinuity is strongly reduced and, below about  $a_H = 0.3 \mu\text{m}$ , the unstable branch disappears and there is a smooth evolution from a very dilute cloud to a less dilute state with an increasing density as  $N$  grows.

For large  $N$  the size of the cloud is remarkably independent of the trap size suggesting that the atoms are in a self-bound configuration. We have verified this effect by integrating numerically the time dependent GP equation and by studying the dynamics of the cloud when the external trap

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is switched off. The condensate expands when  $N < N_{cl} \simeq 234$ . For larger  $N$  the condensate oscillates around the minimum of the energy  $\mathcal{E}(\sigma)$  in absence of external trap (see also next section). Thus, within our representation of the effective interatomic interaction, 234 is the minimum number needed to get a self-bound low density cloud of  ${}^7\text{Li}$  atoms. Its average density is about  $10^{16}$  at/cm<sup>3</sup>. This self-bound state might have a rather short life time due to recombination precesses.

### 3. CONDENSATE COLLECTIVE OSCILLATIONS

The condensate undergoes density oscillations around the minimum  $\sigma_0$  of the energy  $\mathcal{E}(\sigma)$ . In the local case, it has been shown that the monopole collective oscillation  $\omega \rightarrow 0$  as  $N \rightarrow N_c$  with a  $1/4$  power law<sup>6</sup>. It is interesting to analyze what happens by including non locality. We use two schemes: the numerical integration of the time-dependent GP equation and an approximate analytical study of the small oscillations around the minimum of the energy function of the condensate.

Following Ref. 6, we associate with the collective motion a kinetic energy of the form

$$\bar{T} = \frac{1}{2}mN\dot{r}^2 = \frac{3}{4}\frac{N\hbar}{\omega_0}\dot{\sigma}^2, \quad (3)$$

where  $\sigma$  is again the standard deviation of the Gaussian trial wavefunction in units of the harmonic oscillator length  $a_H$ . The dynamics of the collective excitations is determined by  $\bar{T}$  and by the quadratic part of the energy  $\mathcal{E}$  expanded in powers of  $(\sigma - \sigma_0)$ . Some elementary steps lead to a remarkably simple expression for the monopole frequency. In the local case we find

$$\omega = \omega_0 \left[ 5 - \sigma_0^{-4} \right]^{1/2}, \quad (4)$$

where  $\sigma_0$  is related to  $N$  by  $N = (\sigma_0^5 - \sigma_0)/\gamma$  with  $\gamma = (2/\pi)^{1/2}a_T/a_H$ . From Eq. (4) we verify that  $\omega = 2\omega_0$  for  $\sigma_0 = 1$ , and  $\omega \rightarrow 0$  for  $\sigma_0 \rightarrow 5^{-1/4}$  (i.e. for  $N \rightarrow N_c$ ). Instead, in the non local case, the frequency reads

$$\begin{aligned} \omega = \omega_0 \left[ 3\sigma_0^{-4} + 1 + N \left( 4\gamma_R\sigma_0^{-5} + \frac{2}{3}\tau_1\sigma_0^{-3} \right. \right. \\ \left. \left. - \frac{2}{3}\chi^2\tau_2g(\chi\sigma_0)(1 + 2\chi^2\sigma_0^2) + \frac{4}{3\sqrt{\pi}}\chi^3\tau_2\sigma_0 \right) \right]^{1/2}, \end{aligned} \quad (5)$$

where  $\sigma_0$  is related to  $N$  and the interaction parameters by Eq. (2).

In Fig. 2 we show the monopole collective frequency of the condensate as a function of  $N$  for 3 traps for non local, local and trap-off cases. This figure can be easily obtained by using Eq. (2), Eq. (4) and Eq. (5). In

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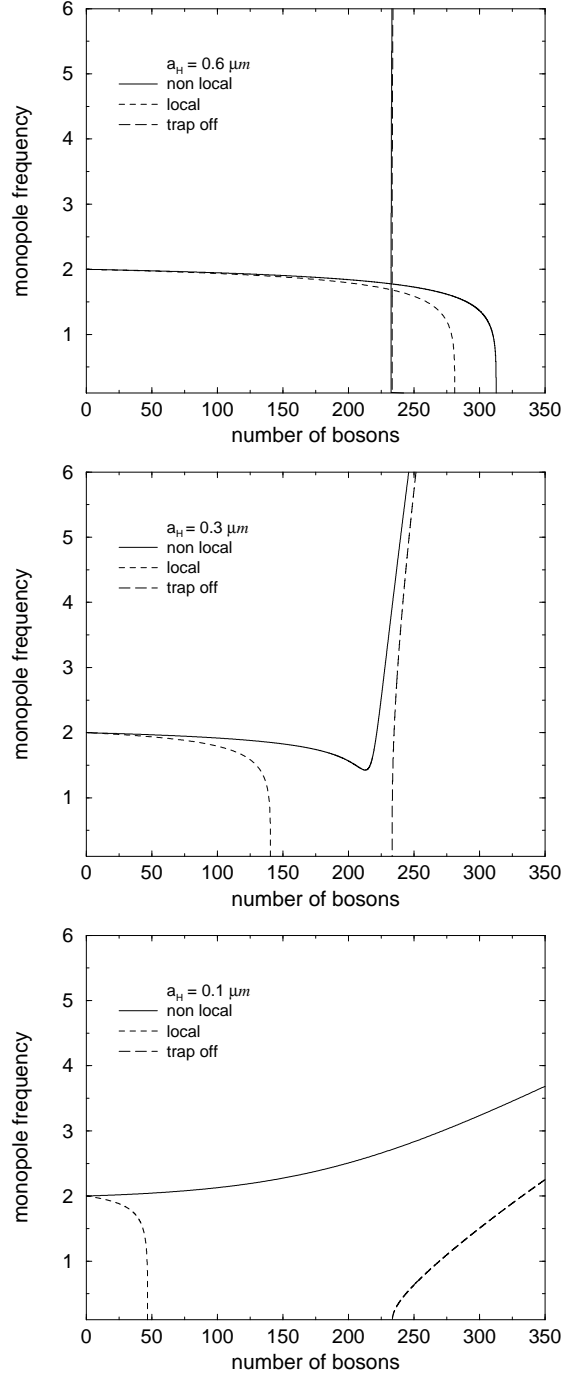


Fig. 2. Monopole frequency of the condensate, in units of trap frequency, as a function of the number of bosons for 3 different traps. From top to bottom:  $a_H = 0.6 \mu m$ ,  $a_H = 0.3 \mu m$ ,  $a_H = 0.1 \mu m$  (respectively,  $\omega_0 = 25.10$  kHz,  $\omega_0 = 100.41$  kHz,  $\omega_0 = 903.67$  kHz). Note that in the trap off case there is a well defined frequency only for  $N > 234$ .

$N$	$\omega$ (numerical)	$\omega$ (analytical)
100	1.91	1.92
200	1.64	1.57
250	6.74	6.62
300	12.69	13.91
500	33.96	37.25
1000	75.70	77.85

Table 1

Numerical and analytical monopole frequency, in units of the trap frequency, for different values of the number  $N$  of bosons.  $a_H = 0.3 \mu\text{m}$ .

the non local case, for the larger trap ( $a_H = 0.6 \mu\text{m}$ ), where there is a reentrant behavior, we see two branches: One branch starts from small  $N$  and corresponds to the larger cloud. The frequency of this branch is very close to the result given by the local approximation. In the second branch  $\omega$  starts from zero at the lowest limit of the reentrant behavior in Fig. 1 and it raises rapidly as  $N^{1/2}$ . For traps of intermediate size ( $a_H = 0.3 \mu\text{m}$  in Fig. 2),  $\omega$  has a dip in the transition region between the low density state and the self bound state. For very small traps, there is only one branch and the frequency increases smoothly with the number of bosons. As discussed in the previous section, when the external trap is switched off the condensate oscillates if  $N > N_{cl} \simeq 234$ . This frequency starts from 0 at  $N_{cl}$  and it approaches the trap-on (non local) frequency by increasing  $N$ . As shown in Table 1, there is good agreement between the variational monopole frequency of Eq. (5) and the numerical one (non local case), obtained by solving the time-dependent GP equation.

To conclude, we observe that it should be kept in mind that in the large  $N$  limit our results are only qualitative because the GP equation itself breaks down and interaction effects are expected to produce a depletion of the condensate when  $\rho|a_T|^3$  is not very small.

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